Multiple Integrals

Double Integrals:- The double integral of a function $f(x,y)$ bounded between the limits of a region R is given by

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dx\,dy \quad \text{or} \quad \iint_R f(x,y)\,dy\,dx$$

Evaluation of double integral:-

Evaluation of double integral depends on the curves bounding the area R.

1) When limits of 'x' are function of y & 'y' has a constant values.

whenever limits of x are functions of y i.e. $\phi_1(y)$ & $\phi_2(y)$ then always consider a horizontal strip PQ s.t.

$$\iint_R f(x,y)\,dx\,dy = \int_c^d \int_{x=\phi_1(y)}^{x=\phi_2(y)} f(x,y)\,dx\,dy$$

2) When limits of 'y' are functions of x & 'x' has constant values:- whenever limits of y are functions of x i.e. $\phi_1(x)$ & $\phi_2(x)$ then always consider a vertical strip PQ s.t.

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_{y=\phi_1(x)}^{y=\phi_2(x)} f(x,y)\,dy\,dx$$

3) When limits of x as well as y are constants.

whenever limits of x as well as y are constants s.t.
$a \le x \le b$ & $c \le y \le d$

$$\iint_R f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Ques: Evaluate $\int_1^2 \int_1^3 xy^2 \, dx \, dy$

Soln:-

$$I = \int_1^2 \int_1^3 xy^2 \, dx \, dy$$

$$= \int_1^2 y^2 \left[\frac{x^2}{2}\right]_1^3 dy$$

$$= \int_1^2 y^2 \frac{1}{2}(9-1) \, dy = \frac{8}{2} \int_1^2 y^2 \, dy$$

$$= 4 \left\{\frac{y^3}{3}\right\}_1^2 = \frac{4}{3}\{8-1\} = \frac{4}{3} \times 7 = \frac{28}{3}.$$

Ques: Evaluate: $\int_1^2 \int_0^x \frac{dx\,dy}{y^2+x^2}$

Soln: $I = \int_1^2 \int_0^x \frac{1}{y^2+x^2} \cdot dy\,dx = \int_1^2 \left[\frac{1}{x} \tan^{-1} \frac{y}{x}\right]_0^x dx$

$$= \int_1^2 \left\{\left(\frac{1}{x} \tan^{-1} \frac{x}{x}\right) - \left(\frac{1}{x} \tan^{-1} 0\right)\right\} dx$$

$$= \int_1^2 \frac{1}{x} \times \frac{\pi}{4} \cdot dx = \frac{\pi}{4} \left\{\log x\right\}_1^2$$

$$= \frac{\pi}{4} (\log 2 - \log 1)$$

$$= \frac{\pi}{4} (\log 2) \qquad (\because \log 1 = 0)$$

Prove that $\int_1^2\int_3^4 (xy+e^y)\,dy\,dx = \int_3^4\int_1^2 (xy+e^y)\,dx\,dy$

Soln:- $L.H.S. = \int_1^2\int_3^4 (xy+e^y)\,dy\,dx$

$= \int_1^2 \left[x\frac{y^2}{2}+e^y\right]_3^4 dx = \int_1^2 \left(\frac{x(4)^2}{2}+e^4\right)-\left(\frac{3^2.x}{2}+e^3\right)dx$

$= \int_1^2 \left(8x+e^4-\frac{9}{2}x-e^3\right)dx = \int_1^2\left(\frac{7}{2}x+e^4-e^3\right)dx$

$= \left[\frac{7}{2}\frac{x^2}{2}+(e^4-e^3)x\right]_1^2$

$= \left[\frac{7}{4}(4-1)+(e^4-e^3)(2-1)\right]$

$L.H.S. = \frac{7\times 3}{4}+(e^4-e^3) = \frac{21}{4}+(e^4-e^3)$

Now, $R.H.S. = \int_3^4\int_1^2 (xy+e^y)\,dx\,dy = \int_3^4\left[y.\frac{x^2}{2}+e^y.x\right]_1^2 dy$

$= \int_3^4 \left\{(2.y+2e^y)-\left(\frac{1}{2}y+e^y\right)\right\}dy$

$= \int_3^4\left(\frac{3}{2}y+e^y\right)dy = \left[\frac{3}{2}\frac{y^2}{2}+e^y\right]_3^4$

$R.H.S = \frac{3}{4}(16-9)+e^4-e^3 = \frac{21}{4}+(e^4-e^3)$

$\Rightarrow L.H.S = R.H.S$

Ques: Evaluate $\int_0^1\int_0^x e^{y/x}\,dy\,dx$

Soln:- $I = \int_0^1\int_0^x e^{y/x}\,dy\,dx$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^x dx = \int_0^1 x(e-1)\,dx$$

$$= (e-1)\int_0^1 x\,dx = (e-1)\left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}(e-1)(1-0)$$

$$= \frac{1}{2}(e-1) \quad \text{Ans}$$

Ques: Evaluate $\int_0^1\int_{x^2}^x (x^2+3y+2)\,dy\,dx$

Soln: $I = \int_0^1\int_{x^2}^x (x^2+3y+2)\,dy\,dx = \int_0^1 \left[x^2y + \frac{3y^2}{2} + 2y\right]_{x^2}^x dx$

$$= \int_0^1 \left[\left(x^3 + \frac{3}{2}x^2 + 2x\right) - \left(x^4 + \frac{3}{2}x^4 + 2x^2\right)\right]dx$$

$$= \int_0^1 \left(x^3 - \frac{5}{2}x^4 - \frac{1}{2}x^2 + 2x\right)dx$$

$$= \left[\frac{x^4}{4} - \frac{5}{2}\cdot\frac{x^5}{5} - \frac{1}{2}\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{4} - \frac{1}{2} - \frac{1}{6} + 1 = \frac{3-6-2+12}{12} = 7/12 \quad \text{Ans}$$

Q: $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx.$

Soln:- $I = \int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dy\,dx$

$$= \int_0^{2a} \left[x^2 y + \frac{y^3}{3}\right]_0^{\sqrt{2ax-x^2}} dx$$

$$= \int_0^{2a} \left\{x^2\sqrt{2ax-x^2} + \frac{1}{3}(2ax-x^2)^{3/2}\right\} dx$$

$$= \int_0^{2a} \left\{x^{5/2}\sqrt{2a-x} + \frac{1}{3}x^{3/2}(2a-x)^{3/2}\right\} dx \quad \text{--- (1)}$$

Substitute $x = 2a\sin^2\theta$

s.t $dx = 4a\sin\theta.\cos\theta\, d\theta$

when $x = 0 \Rightarrow \theta = 0$

when $x = 2a \Rightarrow 2a = 2a\sin^2\theta$

$\Rightarrow \theta = \pi/2$

$\therefore$ (1) $\Rightarrow$ $I = \int_0^{\pi/2} (2a\sin^2\theta)^{5/2}\sqrt{2a-2a\sin^2\theta} + \frac{1}{3}(2a\sin^2\theta)^{3/2}.$

$(2a-2a\sin^2\theta)^{3/2}\Big\} \times$

$4a\sin\theta.\cos\theta\, d\theta$

$$= \int_0^{\pi/2} (2a)^{5/2}.\sin^5\theta.(2a)^{1/2}\sqrt{(1-\sin^2\theta)} + \frac{1}{3}(2a)^{3/2}\sin^3\theta\,(2a)^{3/2}(1-\sin^2\theta)^{3/2}$$

$\times\ 4a\sin\theta.\cos\theta\, d\theta$

$$= \int_0^{\pi/2} \left[(2a)^3.\sin^5\theta.\cos\theta + \frac{1}{3}(2a)^3.\sin^3\theta.\cos^3\theta\right] 4a\sin\theta.\cos\theta\, d\theta$$

$$= 32a^4 \int_0^{\pi/2} \left[\sin^5\theta \cdot \cos\theta + \frac{1}{3}\sin^3\theta \cdot \cos^3\theta\right] \sin\theta \cdot \cos\theta \, d\theta$$

$$= 32a^4 \int_0^{\pi/2} \left(\sin^6\theta \cdot \cos^2\theta + \frac{1}{3}\sin^4\theta \cdot \cos^4\theta\right) d\theta$$

$$= 32a^4 \left\{ \frac{5 \cdot \cancel{3}^{1} \cdot 1 \cdot 1}{8 \cdot \cancel{6}_2 \cdot 4 \cdot 2} \times \frac{\pi}{2} + \frac{1}{\cancel{3}} \; \frac{\cancel{3} \times 1 \times \cancel{3}^{1} \times 1}{8 \cdot \cancel{6}_2 \cdot 4 \cdot 2} \times \frac{\pi}{2} \right\}$$

Using property $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{[(p-1)(p-3) \cdots 1][(q-1)(q-3) \cdots 1]}{(p+q)(p+q-2) \cdots 2} \times \frac{\pi}{2}$

$$= 32a^4 \left[\frac{5\pi}{256} + \frac{\pi}{256} \right]$$

$$= \cancel{32}^{1} a^4 \left(\frac{6\pi}{\cancel{256}_{\cancel{32}_8}} \right) = \frac{6\pi a^4}{8} = \frac{3\pi a^4}{4} \text{ Ans}$$

Q: $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx$

Soln:- $I = \int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\, dy\, dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_x^{\sqrt{x}} dx$

$$= \int_0^1 \cancel{\left[x^2 \cdot \sqrt{x} + \frac{(\sqrt{x})^3}{3} - 0\right] dx} = \int_0^1 \cancel{\left(x^{5/2} + \frac{x^{3/2}}{3}\right) dx}$$

$$= \int_0^1 \left\{\left(x^2 \cdot x^{1/2} + \frac{(\sqrt{x})^3}{3}\right) - \left(x^3 + \frac{x^3}{3}\right)\right\} dx$$

$$= \int_0^1 \left(x^{5/2} + \frac{x^{3/2}}{3} - \frac{4}{3}x^3 \right) dx$$

$$= \left\{ \frac{x^{5/2+1}}{5/2+1} + \frac{x^{3/2+1}}{3\left(\frac{3}{2}+1\right)} - \frac{4}{3}\left(\frac{x^{3+1}}{3+1}\right) \right\}_0^1$$

$$= \left[\frac{2}{7} x^{7/2} + \frac{2}{15} x^{5/2} - \frac{1}{3} x^4 \right]_0^1$$

$$= \left[\frac{2}{7} + \frac{2}{15} - \frac{1}{3} \right\} = \frac{30+14-35}{105} = \frac{44-35}{105} = \frac{9}{105} = \frac{3}{35}$$

Q: Evaluate : $\displaystyle\int_0^1 \int_0^1 \frac{dx\, dy}{\sqrt{(1-x^2)(1-y^2)}}$.

Soln:- $I = \displaystyle\int_0^1 \left\{ \int_0^1 \frac{dx}{\sqrt{1-x^2}} \right\} \frac{1}{\sqrt{(1-y^2)}} \cdot dy$

$$= \int_0^1 \left[\frac{1}{1} \sin^{-1} \frac{x}{1} \right]_0^1 \cdot \frac{1}{\sqrt{1-y^2}} \cdot dy$$

$$= \int_0^1 \left(\frac{\pi}{2} \right) \frac{1}{\sqrt{1-y^2}} \cdot dy$$

$$= \frac{\pi}{2} \cdot \int_0^1 \frac{1}{\sqrt{1-y^2}} \cdot dy = \frac{\pi}{2} \left[\frac{1}{1} \cdot \sin^{-1} \frac{y}{1} \right]_0^1$$

$$= \frac{\pi}{2} \left\{ \sin^{-1} 1 - \sin^{-1} 0 \right\} = \frac{\pi}{2} \times \left(\frac{\pi}{2} - 0 \right)$$

$$= \frac{\pi^2}{4} \text{ Ans}$$

Q: Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$

Soln: $I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{1+x^2+y^2} \cdot dy\,dx$

$$= \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{1}{(\sqrt{1+x^2})^2+y^2} \cdot dy \cdot dx$$

$$= \int_0^1 \left\{ \frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right]_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \left(\tan^{-1} 1 - \tan^{-1} 0 \right) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} \cdot dx = \frac{\pi}{4} \int_0^1 \frac{1}{\sqrt{1+x^2}} dx$$

$$= \frac{\pi}{4} \left\{ \log\left(x + \sqrt{1+x^2}\right) \right]_0^1$$

$$= \frac{\pi}{4} \left\{ \log(1+\sqrt{2}) - \log 1 \right\} \qquad (\because \log 1 = 0)$$

$$= \frac{\pi}{4} \log(1+\sqrt{2}) \quad \text{Ans}$$

Q: Evaluate : $\int_0^1\int_0^1 \frac{dx\,dy}{(1+x^2)(1+y^2)}$.

Soln:-

$$I = \int_0^1 \frac{1}{1+y^2}\cdot\left\{\int_0^1 \frac{1}{1+x^2}\cdot dx\right\}dy$$

$$= \int_0^1 \frac{1}{1+y^2}\left[\frac{1}{1}\tan^{-1}\frac{x}{1}\right]_0^1 dy$$

$$= \int_0^1 \frac{1}{1+y^2}\cdot(\tan^{-1}1-0)\,dy$$

$$= \int_0^1 \frac{1}{1+y^2}\cdot\frac{\pi}{4}\cdot dy = \frac{\pi}{4}\int_0^1 \frac{1}{1+y^2}\,dy$$

$$= \frac{\pi}{4}\left\{\tan^{-1}y - \cancel{0}\right\}_0^1 = \frac{\pi}{4}\left(\tan^{-1}1-0\right\} = \frac{\pi}{4}\times\frac{\pi}{4}$$

$$= \frac{\pi^2}{16} \text{ Ans}$$

Q: Evaluate: $\int_0^{a\sqrt{3}}\int_0^{\sqrt{x^2+a^2}} \frac{x\,dy\,dx}{a^2+x^2+y^2}$

Soln:- Let $I = \int_0^{a\sqrt{3}} x\left\{\int_0^{\sqrt{x^2+a^2}} \frac{1}{(\sqrt{a^2+x^2})^2+y^2}\cdot dy\right\}dx$

$$= \int_0^{a\sqrt{3}} x\left\{\frac{1}{\sqrt{a^2+x^2}}\cdot\tan^{-1}\frac{y}{\sqrt{a^2+x^2}}\right\}_0^{\sqrt{x^2+a^2}} dx$$

$$= \int_0^{a\sqrt{3}} x\left\{\frac{1}{\sqrt{a^2+x^2}}\cdot(\tan^{-1}1-0)\right\}dx$$

$$= \frac{\pi}{4}\int_0^{\sqrt{3}a} \frac{x}{\sqrt{a^2+x^2}} \cdot dx$$

$$= \frac{\pi}{2\times 4}\int_0^{\sqrt{3}a} (2x)\cdot(a^2+x^2)^{-1/2}\, dx$$

$$= \frac{\pi}{8}\left\{\frac{(a^2+x^2)^{1/2}}{1/2}\right\}_0^{\sqrt{3}a}$$

Using $\int f'(x) f^n(x) dx = \frac{f^{n+1}(x)}{n+1}$

$$= \frac{2\pi}{8}\left\{(a^2+3a^2)^{1/2} - (a^2)^{1/2}\right\}$$

$$= \frac{\pi}{4}\left\{2a - a\right\} = \frac{\pi a}{4} \text{ Ans}$$

Q: $\int_0^1 \int_0^{\sqrt{1-x^2}} 4xy\, e^{x^2}\, dy\, dx$

Soln:- $I = 4\int_0^1 x e^{x^2}\left\{\int_0^{\sqrt{1-x^2}} y\, dy\right\} dx$

$$= 4\int_0^1 x e^{x^2}\left\{\frac{y^2}{2}\right\}_0^{\sqrt{1-x^2}} dx$$

$$= 2\int_0^1 x e^{x^2}(1-x^2)\, dx$$

Substituting; $x^2 = t \Rightarrow 2x\,dx = dt$

s.t. at $x=0$; $t=0$

& at $x=1$; $t=1$

$$I = \int_0^1 (2x)\, e^{x^2} (1-x^2)\, dx$$

$$= \int_0^1 e^t (1-t)\, dt = \left[(1-t)e^t - (-1)e^t\right]_0^1$$

$$= (0+e) - (e^0 + e^0)$$

$$I = (e-2) \quad \text{Ans}$$

Q: Evaluate: $\int_0^\infty \int_0^\infty e^{-x^2(1+y^2)}\, x\, dx\, dy$

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Ques: Evaluate: $\displaystyle\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\,.\,dx\,dy$

Soln:
$$I = \int_0^a \left\{ \int_0^{\sqrt{a^2-y^2}} \sqrt{(a^2-y^2)-x^2}\,dx \right\} dy$$

$$= \int_0^a \left[\frac{x\sqrt{a^2-y^2-x^2}}{2} + \frac{(a^2-y^2)}{2} \sin^{-1}\frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left\{ \frac{\sqrt{a^2-y^2}\,.\,\sqrt{a^2-y^2-a^2+y^2}}{2} + \frac{(a^2-y^2)}{2} \sin^{-1}\frac{\sqrt{a^2-y^2}}{\sqrt{a^2-y^2}} - 0 \right\} dy$$

$$= \int_0^a \left\{ \frac{\sqrt{a^2-y^2}\,.\,0}{2} + \frac{(a^2-y^2)}{2} \sin^{-1} 1 \right\} dy$$

$$= \int_0^a \frac{(a^2-y^2)}{2}\,.\,\frac{\pi}{2}\,.\,dy = \frac{\pi}{4}\int_0^a (a^2-y^2)\,dy$$

$$= \frac{\pi}{4}\left\{ a^2y - \frac{y^3}{3} \right\}_0^a$$

$$= \frac{\pi}{4}\left\{ \left(a^3 - \frac{a^3}{3}\right) - 0 \right\}$$

$$= \frac{\pi}{4}\left\{ \frac{2a^3}{3} \right\} = \frac{\pi a^3}{\cancel{12}\,6} \quad \text{Ans}$$

Ques: Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$

Soln:- Given equation is $x+y \le 1$

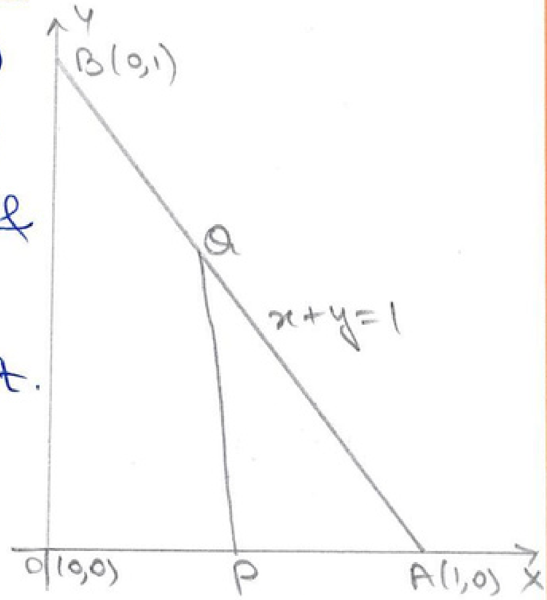
as $x=0$; $y \le 1 \Rightarrow (0,1)$

& as $y=0$; $x \le 1 \Rightarrow (1,0)$

$\therefore$ It is a straight line AB & the area is ΔOAB. Let us consider a vertical strip PQ s.t.

at P: $y=0$

at Q; $y = 1-x$



$\therefore R = \{(x,y) : 0 \le x \le 1 \ \& \ 0 \le y \le 1-x\}$

$\therefore I = \text{Area of } \Delta OAB = \iint_R (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left\{x^2(1-x) + \frac{1}{3}(1-x)^3\right\} dx$$

$$= \int_0^1 \left[(x^2 - x^3) + \frac{1}{3}(1-x)^3\right] dx = \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\frac{(1-x)^4}{-4}\right]_0^1$$

$$= \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{12}(0)\right) - \left(0 - 0 - \frac{1}{12}(1)^4\right) = \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12} = \frac{2}{12} = \frac{1}{6}$$

Ques: Evaluate $\iint (x^2+y^2)\,dx\,dy$ over the region in the positive quadrant for which $x+y \le 1$

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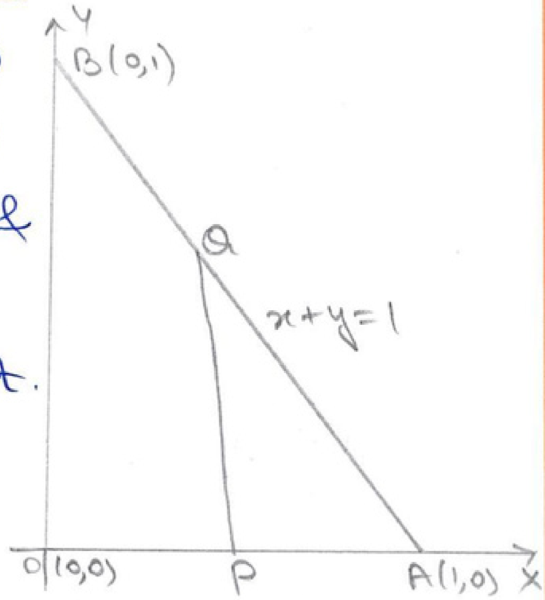

at P: $y=0$

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$\therefore I = \text{Area of } \Delta OAB = \iint_R (x^2+y^2)\,dy\,dx$

$$= \int_0^1 \int_0^{1-x} (x^2+y^2)\,dy\,dx = \int_0^1 \left[x^2 y + \frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left\{x^2(1-x) + \frac{1}{3}(1-x)^3\right\} dx$$

$$= \int_0^1 \left[(x^2-x^3) + \frac{1}{3}(1-x)^3\right] dx = \left[\frac{x^3}{3} - \frac{x^4}{4} + \frac{1}{3}\frac{(1-x)^4}{-4}\right]_0^1$$

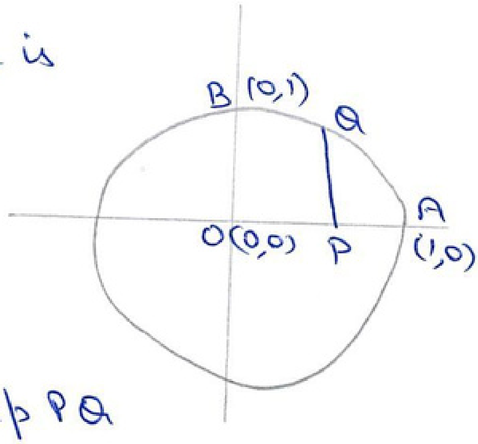
$$= \left(\frac{1}{3} - \frac{1}{4} - \frac{1}{12}(0)\right) - \left(0 - 0 - \frac{1}{12}(1)^4\right) = \frac{1}{3} - \frac{1}{4} + \frac{1}{12}$$

$$= \frac{4-3+1}{12} = \frac{2}{12} = \frac{1}{6}$$

Q: Evaluate $\iint x^2y^2\,dx\,dy$ over the circle $x^2+y^2=1$.

Soln:- Here the given eqn. of circle is

$x^2+y^2=1$; i.e. Centre is $(0,0)$ & radius 1.



Since, the circle is symmetric

$\therefore$ Area of circle $= 4\times$(Area OAB)

Let us consider a vertical strip PQ s.t. at P; $y=0$

at Q; $y=\sqrt{1-x^2}$ $\quad \{ x^2+y^2=1 \Rightarrow y^2=1-x^2 \Rightarrow y=\sqrt{1-x^2}$

$\therefore R=\{(x,y);\ 0\le x\le 1;\ 0\le y\le\sqrt{1-x^2}\}$

$$\therefore I=\iint_R x^2y^2\,dx\,dy = 4\int_0^1\int_0^{\sqrt{1-x^2}} x^2y^2\,dy\,dx$$

$$=4\int_0^1 x^2.\left[\frac{y^3}{3}\right]_0^{\sqrt{1-x^2}}dx = \frac{4}{3}\int_0^1 x^2(1-x^2)^{3/2}\,dx \quad \text{—— (1)}$$

Substituting $x=1.\sin\theta \Rightarrow dx=\cos\theta\,d\theta$

when $x=0 \Rightarrow \theta=0$ & when $x=1 \Rightarrow \theta=\pi/2$

$$I=\frac{4}{3}\int_0^{\pi/2}\sin^2\theta.(1-\sin^2\theta)^{3/2}\cos\theta\,d\theta$$

$$=\frac{4}{3}\int_0^{\pi/2}\sin^2\theta.\cos^4\theta\,d\theta$$

$$=\frac{4}{3}\times\frac{(2-1)\times(4-1)(4-3)}{6\times4\times2}\times\frac{\pi}{2}$$

$$=\frac{\cancel{4}}{\cancel{3}}\times\frac{1\times\cancel{3}\times}{6\times\cancel{4}\times2}\frac{\pi}{2}$$

$$=\frac{\pi}{24}\ \text{Ans}$$

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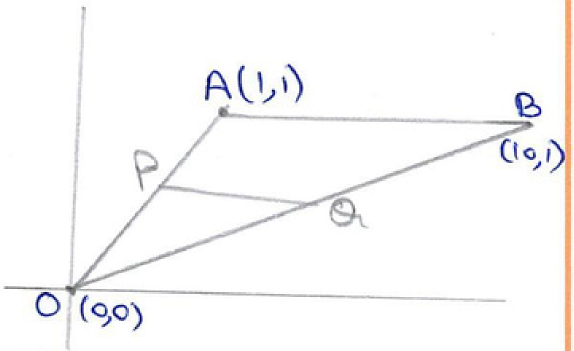
Q: Evaluate $\iint_S \sqrt{xy-y^2}\, dx\, dy$; where S is a triangle with vertices (0,0), (10,1) & (1,1).

Soln:- In the triangle OAB;



Eqn. of line OA;

$$y-0 = \frac{1-0}{1-0}(x-0)$$

$$\Rightarrow \boxed{y=x} \quad \text{—— (1)}$$

Eqn. of line OB; $y-0 = \frac{1-0}{10-0}(x-0)$

$$y = \frac{1}{10}x \Rightarrow \boxed{x=10y} \quad \text{—— (2)}$$

Eqn. of line AB; $y-1 = \frac{1-1}{10-1}(x-1)$

$$y-1=0 \Rightarrow \boxed{y=1} \quad \text{—— (3)}$$

Let us consider a horizontal strip PQ s.t. at P; $x=y$ & at Q; $x=10y$

$\therefore \quad S = \{(x,y);\ 0 \le y \le 1;\ y \le x \le 10y\}$

$$\therefore I = \int_0^1 \int_y^{10y} \sqrt{xy-y^2}\, .dx.dy$$

$$= \int_0^1 \left[\frac{(xy-y^2)^{3/2}}{3/2 \cdot y}\right]_y^{10y} dy = \frac{2}{3}\int_0^1 \frac{1}{y}\left\{(10y^2-y^2)^{3/2} - (y^2-y^2)^{3/2}\right\} dy$$

$$= \frac{2}{3}\int_0^1 \frac{1}{y}(9y^2)^{3/2} dy = \frac{2}{3}\int_0^1 27.y^2 dy$$

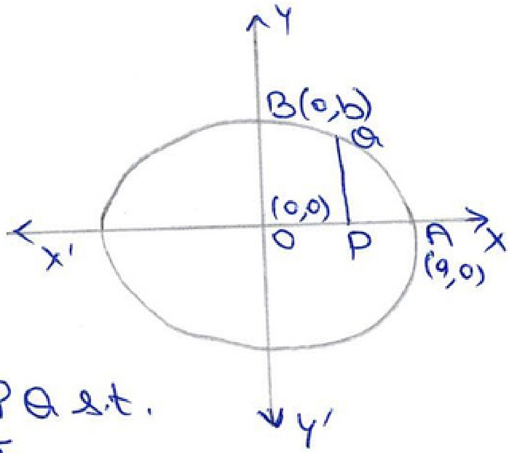
$$= 18\int_0^1 y^2 dy = 18\left[\frac{y^3}{3}\right]_0^1 = 6(1-0)$$

$$= 6 \text{ Ans}$$

Q: Evaluate $\iint_R y\,dx\,dy$; where R is the region bounded by ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in 1st Quadrant.

Soln. In the given eqn;

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{——— (1)}$$



Sub $x=0$; $y=b \Rightarrow B(0,b)$

Sub $y=0$; $x=a \Rightarrow A(a,0)$

In first Quadrant; OAB;

let us consider a vertical strip PQ s.t.

at P; $y=0$ at Q; $y=\frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore R=\left\{(x,y) : 0\le x\le a;\ 0\le y\le \frac{b}{a}\sqrt{a^2-x^2}\right\}$$

$$\therefore I=\iint_R y\,dx\,dy=\int_0^a\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} y\,dy\,dx$$

$$=\int_0^a\left[\frac{y^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}}dx=\frac{1}{2}\int_0^a\frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$=\frac{b^2}{2a^2}\int_0^a(a^2-x^2)\,dx=\frac{b^2}{2a^2}\left\{a^2x-\frac{x^3}{3}\right]_0^a$$

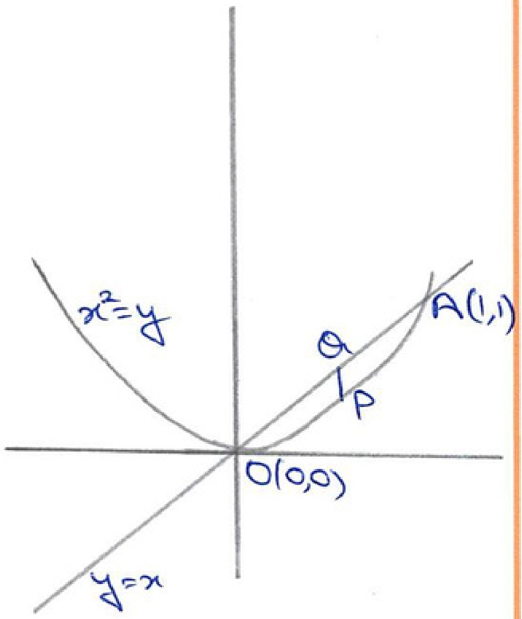
$$=\frac{b^2}{2a^2}\left\{a^3-\frac{a^3}{3}\right\}=\frac{b^2}{2a^2}\times\frac{2a^3}{3}=\frac{1}{3}ab^2 \text{ Ans}$$

Q: Evaluate: $\iint xy(x+y)\,dy\,dx$ over the area between $y=x^2$ & $y=x$.

Soln. The given equations are

$$y=x^2 \quad \text{—(1)} \quad \& \quad y=x \quad \text{—(2)}$$

Now; Eqn (1) is an upward parabola with vertex at O & Eqn (2) is a straight line passing through Origin.



To find point's of Intersection

from eqn (1) & (2)

$$x^2=x \Rightarrow x^2-x=0$$
$$x(x-1)=0 \Rightarrow x=0 \text{ or } x=1$$

If $x=0$; $y=0 \Rightarrow O(0,0)$

If $x=1$; $y=1 \Rightarrow A(1,1)$

Let us consider a vertical strip PQ s.t.

at P; $y=x^2$ at Q; $y=x$

$$\therefore R=\{(x,y): 0\le x\le 1;\ x^2\le y\le x\}$$

$$\therefore I=\iint_R xy(x+y)\,dy\,dx = \int_0^1\int_{x^2}^{x}(x^2y+xy^2)\,dy\,dx$$

$$=\int_0^1\left[\frac{x^2y^2}{2}+\frac{xy^3}{3}\right]_{x^2}^{x}dx = \int_0^1\left\{\left(\frac{x^4}{2}+\frac{x^4}{3}\right)-\left(\frac{x^6}{2}+\frac{x^7}{3}\right)\right\}dx$$

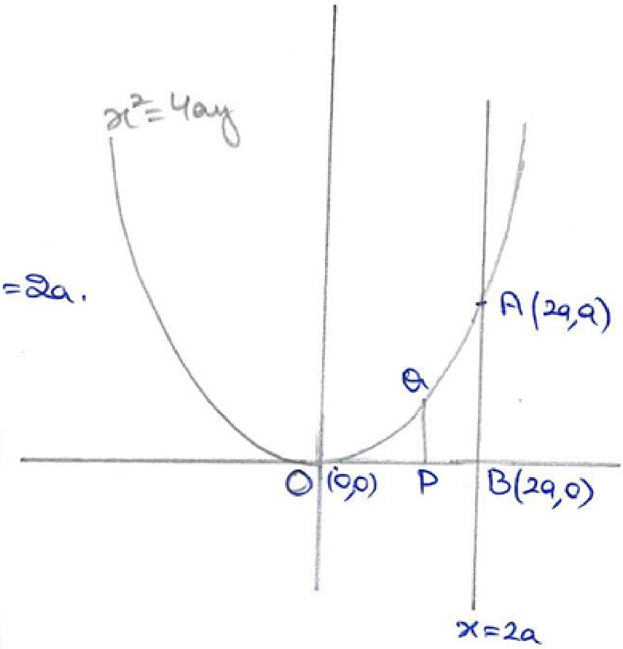
$$=\int_0^1\left(\frac{5}{6}x^4-\frac{1}{2}x^6-\frac{1}{3}x^7\right)dx = \left[\frac{5}{6}\cdot\frac{x^5}{5}-\frac{1}{2}\frac{x^7}{7}-\frac{1}{3}\frac{x^8}{8}\right]_0^1$$

$$=\frac{1}{6}-\frac{1}{14}-\frac{1}{24}=\frac{28-12-7}{168}=\frac{9}{168}=\frac{3}{56}\ \text{Ans}$$

Q: Evaluate $\iint_A xy\,dx\,dy$; where A is the domain bounded by x-axis, ordinate $x = 2a$ & the curve $x^2 = 4ay$.

Soln:- The given equation is

$x^2 = 4ay$; which is an upward parabola with vertex at Origin O(0,0). & the ordinate $x = 2a$.



To find points of Intersection

$x^2 = 4ay$ — (1) & $x = 2a$ — (2)

from (1) & (2); $(2a)^2 = 4ay$

$4a^2 = 4ay \Rightarrow \boxed{y = a}$

$\therefore$ A(2a, a)

$\therefore$ Reqd region = OAP region.

Let us consider a vertical strip PQ s.t. at P; $y = 0$ at Q; $y = x^2/4a$.

$\therefore R = \{(x,y) : 0 \le x \le 2a;\ 0 \le y \le x^2/4a\}$

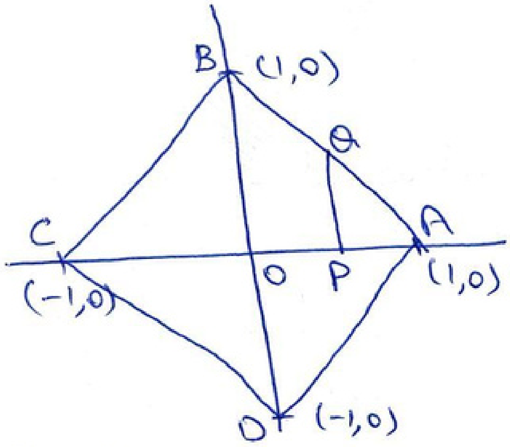
$$I = \int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx = \int_0^{2a} x\left[\frac{y^2}{2}\right]_0^{x^2/4a} dx$$

$$= \frac{1}{2}\int_0^{2a} x\left(\frac{x^4}{16a^2} - 0\right)dx = \frac{1}{32a^2}\int_0^{2a} x^5\,dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a} = \frac{1}{32a^2 \times 6}\left((2a)^6 - 0\right)$$

$$= \frac{\cancel{64}^{\,1} a^6}{\cancel{32} \times \cancel{6}_{\,3} \times a^2} = \frac{a^4}{3} \text{ Ans}$$

Q: Sketch the region of integration and evaluate $\iint_R (y-2x^2)\,dx\,dy$ where R is the region inside the square $|x|+|y|=1$.



Soln:- The given equation is

$$|x|+|y|=1 \quad \text{—— (1)}$$

sub $y=0$; $|x|=1 \Rightarrow x=\pm 1$

sub $x=0$; $|y|=1 \Rightarrow y=\pm 1$

$\therefore$ vertices are $(0,1)\ (0,-1),\ (1,0),\ (-1,0)$.

$\therefore$ Reqd Area = Area of square ABCD

= 4 × Area OAB.

Let us consider a vertical strip PQ s.t.

at P: $y=0$ at Q: $y=1-x$

$\therefore R=\{(x,y): 0\le x\le 1\ ;\ 0\le y\le 1-x\}$

$$\text{Reqd Area} = 4\int_0^1\int_0^{1-x}(y-2x^2)\,dy\,dx$$

$$=4\int_0^1\left[\frac{y^2}{2}-2x^2y\right]_0^{1-x}dx$$

$$=4\int_0^1\left\{\frac{(1-x)^2}{2}-2x^2(1-x)\right\}dx$$

$$=4\int_0^1 \tfrac{1}{2}(1+x^2-2x)-2x^2+2x^3\}\,dx$$

$$= \frac{4}{2}\int_0^1 (1 + x^2 - 2x - 4x^2 + 4x^3)\,dx$$

$$= 2\int_0^1 (1 - 3x^2 - 2x + 4x^3)\,dx = 2\left[x - \cancel{3}\frac{x^3}{\cancel{3}} - \cancel{2}\frac{x^2}{\cancel{2}} + \cancel{4}\frac{x^4}{\cancel{4}}\right]_0^1$$

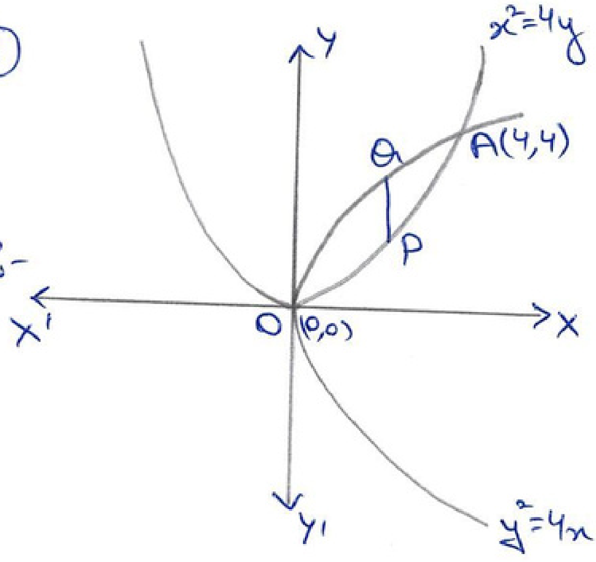
$$= 2\{1 - 1 + 1 - 1\} = 2(0) = 0 \text{ Ans}$$

Q: Evaluate $\iint_R y\,dx\,dy$; where R is the region bounded by the parabola $y^2 = 4x$ & $x^2 = 4y$.

Soln:- The given equations of parabola's are

$y^2 = 4x \Rightarrow x = y^2/4$ — ①

& $x^2 = 4y$ — ②



To find points of Intersection :-

from eqn ① & ②

$$\left(\frac{y^2}{4}\right)^2 = 4y$$

$$y^4 = 64y$$

$$y^4 - 64y = 0 \Rightarrow y(y^3 - 64) = 0$$

$$\Rightarrow y = 0;\ y = 4$$

when $y = 0 \Rightarrow x = 0$ ∴ O(0,0)

when $y = 4 \Rightarrow x = 4$ ∴ A(4,4)

Let us consider a vertical strip PQ s.t.

at P; $y = x^2/4$ at Q; $y = \sqrt{4x} = 2\sqrt{x}$

$\therefore R = \{(x,y) : 0 \le x \le 4;\ \frac{x^2}{4} \le y \le 2\sqrt{x}\}$

$\therefore$ Reqd Area $= \int_0^4 \int_{x^2/4}^{2\sqrt{x}} y\, dy\, dx$

$= \int_0^4 \left[\frac{y^2}{2}\right]_{x^2/4}^{2\sqrt{x}} dx$

$= \frac{1}{2}\int_0^4 \left\{(2\sqrt{x})^2 - \left(\frac{x^2}{4}\right)^2\right\} dx$

$= \frac{1}{2}\int_0^4 \left(4x - \frac{x^4}{16}\right) dx$

$= \frac{1}{2}\left[\frac{4x^2}{2} - \frac{x^5}{16\times 5}\right]_0^4$

$= \frac{1}{2}\left\{2(4)^2 - \frac{(4)^5}{16\times 5} - 0\right\}$

$= \frac{1}{2}\left\{32 - \frac{64}{5}\right\} = \frac{1}{2}\left(\frac{160-64}{5}\right)$

$= \frac{1}{2}\left(\frac{96}{5}\right) = \frac{48}{5}$ Ans

Q: Evaluate $\iint (x+y)^2\, dx\, dy$ over the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Soln:- The given eqn. of ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ —— (1)

at $x = 0$; $y = \pm b \Rightarrow (0,b)\ (0,-b)$

at $y = 0$; $x = \pm a \Rightarrow (a,0), (-a,0)$

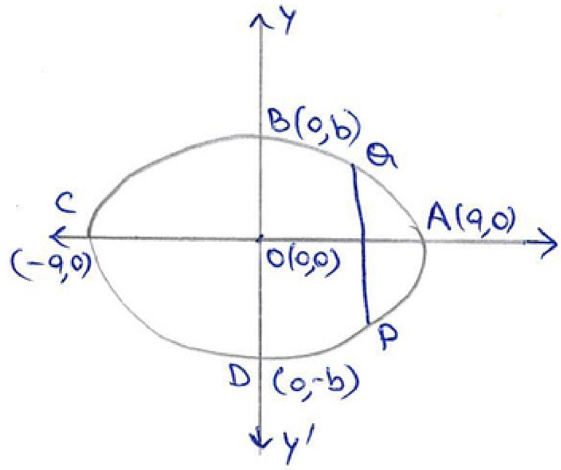
Reqd area = Area of ABCD

from equation ①;

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2)$$

$$y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$$



Since; $f(x,y) = (x+y)^2 = x^2 + y^2 + 2xy$; the first two terms x^2+y^2 is symmetric where as the last term is not symm. due to presence of x & y!

∴ Let us take a vertical strip PQ s.t.

at P: $y = -\frac{b}{a}\sqrt{a^2-x^2}$ & at Q: $y = \frac{b}{a}\sqrt{a^2-x^2}$

$$\therefore y: -\frac{b}{a}\sqrt{a^2-x^2} \longrightarrow \frac{b}{a}\sqrt{a^2-x^2} \quad \& \; x: -a \longrightarrow a$$

$$\therefore \text{Reqd Area} = \int_{-a}^{a} \int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} \{(x^2+y^2) + 2xy\}\, dy\, dx$$

By using the property $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx ; & \text{if } f(x) \text{ is even} \\ 0 ; & \text{if } f(x) \text{ is odd.} \end{cases}$

$$= 2\int_{-a}^{a} \int_{0}^{\frac{b}{a}\sqrt{a^2-x^2}} (x^2+y^2)\,dy\,dx + 0$$

$$= 2\int_{-a}^{a} \left[x^2 y + \frac{y^3}{3}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= 2\int_{-a}^{a}\left\{x^2\cdot\frac{b}{a}\sqrt{a^2-x^2}+\frac{b^3}{3a^3}\cdot(a^2-x^2)^{3/2}\right\}dx$$

$$I = 4\int_0^a\left\{\frac{b}{a}x^2\sqrt{a^2-x^2}+\frac{b^3}{3a^3}(a^2-x^2)^{3/2}\right\}dx$$

substituting $x = a\sin\theta \Rightarrow dx = a\cos\theta\, d\theta$
s.t. as $x=0 \Rightarrow \theta = 0$ & $x = a \Rightarrow \theta = \pi/2$

$$I = 4\int_0^{\pi/2}\left\{\frac{b}{a}a^2\sin^2\theta\sqrt{a^2-a^2\sin^2\theta}+\frac{b^3}{3a^3}(a^2-a^2\sin^2\theta)^{3/2}\right\}a\cos\theta\, d\theta$$

$$= 4\int_0^{\pi/2}\left(ba^2\sin^2\theta\cos\theta+\frac{b^3}{3}\cos^3\theta\right)a\cos\theta\, d\theta$$

$$= 4\left\{a^3b\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta+\frac{\cancel{9}a}{3}b^3\int_0^{\pi/2}\cos^4\theta\, d\theta\right\}$$

$$= 4\left\{a^3b\times\frac{1\times1}{4\times2}\times\frac{\pi}{2}+\frac{ab^3}{3}\times\frac{3\times1}{4\times2}\times\frac{\pi}{2}\right\}$$

$$= \left(\frac{\cancel{4}a^3b}{\cancel{4}\times2}+\frac{\cancel{4}\times\cancel{3}ab^3}{\cancel{3}\times\cancel{4}\times2}\right)\frac{\pi}{2}$$

$$= \frac{\pi}{4}ab(a^2+b^2)$$

$$I = ab(a^2+b^2)\frac{\pi}{4} \quad \text{Ans}$$